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Dirichlet series with periodic coefficients in function fields

Yoshinori Hamahata
Department of Applied Mathematics
Okayama University of Science

1. INTRODUCTION

Sarvadaman Chowla [7] proved that if p is an odd prime, then $\cot(2\pi j/p)$ ($j = 1, \dots, (p-1)/2$) are linearly independent over the field of rational numbers. This result follows from the non-vanishing of the Dirichlet L -series $L(s, \chi)$ at $s = 1$, when χ is a Dirichlet character with $\chi(-1) = -1$. For another proof of Chowla's theorem, we refer the reader to [1, 2, 10, 12]. We note that Chowla's result was generalized in [13, 16].

In [6], which was written in 1964, Chowla raised the following question:

We assume that p is a prime and f is a non-zero function defined on the integers with integer values and period p . Then, does the infinite series

$$\sum_{n=1}^{\infty} \frac{f(n)}{n}$$

never vanish?

The Chowla question is valid whenever the series converges, which is equivalent to the condition $\sum_{a=1}^p f(a) = 0$. Concerning this question, Baker, Birch, and Wirsing [4] proved the following:

Theorem 1 (Baker–Birch–Wirsing [4, 13]). *Let m be a positive integer and f a non-zero function defined on the integers with algebraic values and period m such that*

- (i) $f(r) = 0$ if $1 < \gcd(r, m) < m$.
- (ii) *The m -th cyclotomic polynomial Ψ_m is irreducible over $\mathbb{Q}(f(1), \dots, f(m))$.*

Then,

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0.$$

Let \mathbb{C}_{∞} be the completion of an algebraic closure of the field $\mathbb{F}_q((T^{-1}))$. The Carlitz exponential function $e(z)$, which is defined over \mathbb{C}_{∞} , is given by

$$(1.1) \quad e(z) = z + \sum_{n=1}^{\infty} \frac{z^{q^n}}{(T^{q^n} - T^{q^{n-1}}) \cdots (T^{q^n} - T)}.$$

Then, its reciprocal $c(z) := e(z)^{-1}$ is analogous to $\cot z$. In this report, using $c(z)$, we establish an analog of Chowla's theorem over function fields. As an application, we give an analog of the Baker–Birch–Wirsing theorem about the non-vanishing of Dirichlet series with periodic coefficients at $s = 1$ in the function field setup with a parity condition.

2. SOME FUNCTIONS IN FUNCTION FIELDS

Let \mathbb{F}_q be the finite field with q elements, where q is a power of the prime number p . Let $A = \mathbb{F}_q[T]$ and $K = \mathbb{F}_q(T)$. Let $K_\infty = \mathbb{F}_q((T^{-1}))$ be the completion of K at $\infty = (T^{-1})$, and let \mathbb{C}_∞ be the completion of an algebraic closure \overline{K} of K_∞ . For a ring R , R^* denotes the unit group of R .

2.1. The Carlitz exponential. We denote by $A\{\tau\}$ the twisted polynomial ring whose multiplication is defined by $\tau a = a^q \tau$ ($a \in A$). The \mathbb{F}_q -linear ring homomorphism $\rho : A \rightarrow A\{\tau\}$, defined by $1 \mapsto \tau^0$ and $T \mapsto \rho_T = T\tau^0 + \tau$, is called the Carlitz A -module. With each $N \in A \setminus \{0\}$, ρ associates an additive polynomial $\rho_N(x)$ given by $\rho_N(x) := \rho_N(\tau)(x) \in A[x]$. This is called the Carlitz N -polynomial. For $N \in A \setminus \{0\}$, let $\rho[N] = \{\alpha \in \mathbb{C}_\infty \mid \rho_N(\alpha) = 0\}$ be the set of Carlitz N -torsion points. The set $\rho[N]$ is a cyclic A -module and its generator (as a Carlitz A -module) is called the primitive Carlitz N -torsion point. The minimal polynomial $\Phi_N(x)$ of any primitive N -torsion point over K is called the Carlitz N -th cyclotomic polynomial. The polynomials $\rho_N(x)$ and $\Phi_N(x)$ have degrees $q^{\deg N}$ and $\varphi(N)$, respectively, where $\varphi(N) := \#(A/NA)^*$. For details on these polynomials, we refer the reader to [3]. For the primitive Carlitz N -torsion point λ_N , let $K_N = K(\lambda_N)$ be the field generated over K by adjoining λ_N . If $\sigma \in \text{Gal}(K_N/K)$, then $\sigma(\lambda_N)$ is another primitive Carlitz N -torsion point. Hence, there exists $a \in A$ with $\gcd(a, N) = 1$ such that $\sigma(\lambda_N) = \rho_a(\lambda_N)$. The correspondence $\sigma \mapsto a$ induces the isomorphism $\text{Gal}(K_N/K) \xrightarrow{\sim} (A/NA)^*$ (see [15, Theorem 12.8]).

There exists a unique entire function $e(z)$ over \mathbb{C}_∞ such that for each $a \in A$, we have $\rho_a(e(z)) = e(az)$ (see [9, Chapter 3]). The function $e(z)$ is called the Carlitz exponential. The function denoted by $e(z)$ in the Introduction is exactly the Carlitz exponential. Let L be the set of all zeros of $e(z)$. Then, L is a rank one free A -module (see [9, Corollary 3.2.9]). It is well known that $L = \pi A$ is analogous to $\pi\mathbb{Z}$. Using L , $e(z)$ can be written as

$$(2.1) \quad e(z) = z \prod_{0 \neq l \in L} \left(1 - \frac{z}{l}\right).$$

From (1.1), it holds that $e'(z) = 1$.

2.2. The cotangent function. Let $c(z) := e(z)^{-1}$. Using (1.1) and (2.1), we have

$$(2.2) \quad c(z) = \frac{e'(z)}{e(z)} = \frac{1}{z} + \sum_{0 \neq l \in L} \frac{1}{z + l}.$$

The analogy between $c(z)$ and the usual cotangent function is that $\cot z$ is $\pi\mathbb{Z}$ -periodic and is expressed by

$$\cot z = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z + \pi n} + \frac{1}{z - \pi n} \right);$$

$c(z)$ is L -periodic and is expressed by (2.2).

For a positive integer n , let $P_n(z) = \sum_{l \in L} (z + l)^{-n}$. According to Goss [8], there exists a monic polynomial $G_n(X) \in K[X]$ of degree n such that $P_n(z) =$

$G_n(e(z)^{-1})$. This is called the Goss polynomial. Let $\bar{\pi}$ be a generator of L . Petrov [14] proved that

$$(2.3) \quad D_{n-1}c(\bar{\pi}z) = (-\bar{\pi})^{n-1}P_n(\bar{\pi}z) = (-\bar{\pi})^{n-1}G_n(e(\bar{\pi}z)^{-1}),$$

where D_{n-1} is the $n-1$ -th hyperdifferential operator in z that is discussed by Bosser and Pellarin in [5].

2.3. Goss L -functions. Let A_+ be the set of all monic elements in A and let $M \in A_+$ with $\deg M > 0$. A group homomorphism $\chi : (A/MA)^* \rightarrow \mathbb{C}_\infty^*$ is called a character modulo M . This can be extended to A by

$$\chi(a) = \begin{cases} \chi(a + MA) & \text{if } \gcd(a, M) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

For this χ , let $\bar{\chi}$ be the character defined by

$$\bar{\chi}(a) = \begin{cases} \chi((a + MA)^{-1}) & \text{if } \gcd(a, M) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The Goss L -function of χ is defined by

$$L(s, \chi) = \sum_{a \in A_+} \frac{\chi(a)}{a^s} \quad (s \in \mathbb{N}).$$

This can be thought of as an entire function on the Goss plane $S_\infty := \mathbb{C}_\infty^* \times \mathbb{Z}_p$, where \mathbb{Z}_p is the ring of p -adic integers. Moreover, it has the following Euler product expression

$$L(s, \chi) = \prod_{\substack{P \in A_+ \\ P: \text{irreducible}}} (1 - \chi(P)P^{-s})^{-1}.$$

From this, we have that

$$(2.4) \quad L(s, \chi) \neq 0 \quad (s \in \mathbb{N}).$$

For more details, we refer the reader to the book [9, Chapter 8].

3. THE MAIN THEOREM

Let $M \in A_+$ with $\deg M > 0$ and let R_M be the subset of A_+ defined by

$$R_M = \{a \in A_+ \mid \deg a < \deg M, \gcd(a, M) = 1\}.$$

The main theorem of this paper is the following.

Theorem 2. *Let n be a positive integer. Then, $D_{n-1}c(\bar{\pi}z)|_{z=b/M}$ ($b \in R_M$) are linearly independent over K .*

Remark. 1. In [13, 16], the following result was proved: Let n and m be positive integers with $m > 2$. Let R be a set of $\phi(m)/2$ representatives mod m such that the union $\{R, -R\}$ is a complete set of residues prime to m , where ϕ is the Euler totient function. Then, $(\frac{d}{dz})^{n-1}(\cot \pi z)|_{z=a/m}$ ($a \in R$) are linearly independent over the field of rational numbers. Theorem 2 is an analog of this result. The case $n = 1$ in Theorem 2 is an analog of Chowla's theorem, which was mentioned in the introduction.

2. Hasse [10] proved that for an odd prime p , $\tan(\pi j/p)$ ($j = 1, \dots, (p-1)/2$) are linearly independent over the field of rational numbers. Even though the Carlitz exponential $e(z)$ is analogous to $\tan(z)$, an analog of Hasse's result for $e(z)$ does not hold. In fact, for an irreducible polynomial $P \in A_+$ with $\deg P > 1$, $e(\pi b/P)$ ($b \in A_+$, $\deg b < \deg P$) are linearly dependent over K .

As an application of the above theorem, we have the following, which is an analog of the Baker–Birch–Wirsing theorem (Theorem 1) under a certain condition.

Theorem 3. *Let n be a positive integer and $g : A \rightarrow \overline{K}$ a non-zero function, which is defined on A/MA and then extended to A , such that*

- (i) $g(\zeta a) = \zeta^n g(a)$ ($a \in A, \zeta \in \mathbb{F}_q^*$);
- (ii) $g(a) = 0$ if $\gcd(a, M) > 1$;
- (iii) *The Carlitz M -th cyclotomic polynomial Φ_M is irreducible over K_g , which is the field generated over K by adjoining $\{g(b) \mid b \in A/MA\}$.*

Then,

$$\sum_{a \in A_+} \frac{g(a)}{a^n} \neq 0.$$

Remark. Let k be a positive integer. Okada [13] proved the following: Let f be a non-zero function defined on the integers with algebraic values and period $m > 2$ such that

- (i) f is even or odd according as k is even or odd;
- (ii) $f(n) = 0$ if $\gcd(n, m) > 1$;
- (iii) The m -th cyclotomic polynomial Ψ_m is irreducible over $\mathbb{Q}(f(1), \dots, f(m))$.

Then, $\sum_{n=1}^{\infty} \frac{f(n)}{n^k} \neq 0$.

Theorem 3 is an analog of this result.

From this theorem, we obtain the following.

Theorem 4. *Let n be a positive integer. Let $G = \{\chi : (A/MA)^* \rightarrow \mathbb{C}_{\infty}^*\}$ be the set of all characters modulo M . For $\Lambda = \{\chi \in G \mid \chi(\zeta) = \zeta^n \ (\zeta \in \mathbb{F}_q^*)\}$, let \mathbb{F}_{q^r} be the finite field generated over \mathbb{F}_q by adjoining $\{\chi(b) \mid \chi \in \Lambda, b \in (A/MA)^*\}$. If $\gcd(\varphi(M), r) = 1$, then the Goss L -functions $L(n, \chi)$ ($\chi \in \Lambda$) are linearly independent over K .*

Remark. Let k and m be positive integers with $m > 2$ and $\gcd(m, \phi(m)) = 1$. Let Λ denote the set of all even or odd Dirichlet characters modulo m according as k is even or odd. Then, Okada [13] proved that the Dirichlet L -functions $L(k, \chi)$ ($\chi \in \Lambda$) are linearly independent over the field of rational numbers. Theorem 4 is an analog of this result as well.

4. OUTLINE OF THE PROOFS OF THEOREMS 2, 3, AND 4

4.1. Proof of Theorem 2. We will use two lemmas. The first is a form of the Frobenius determinant relation (see Lang [11, Chapter 21]).

Lemma 5. Let G be a finite abelian group and H a subgroup. Let $\lambda : H \rightarrow \mathbb{C}_\infty^*$ be a character of H and Λ the set of all characters of G given by

$$\Lambda = \{\chi : G \rightarrow \mathbb{C}_\infty^* \mid \chi|_H = \lambda\}.$$

Then, for a \mathbb{C}_∞ -valued function f on G with

$$f(ah) = \lambda(h)f(a) \quad (a \in G, h \in H),$$

we have

$$\det_{b,c \in R} f(b^{-1}c) = \prod_{\chi \in \Lambda} \left(\sum_{a \in R} \bar{\chi}(a) f(a) \right),$$

where R is a complete set of representatives of G/H .

The second lemma connects $D_{n-1}c(\bar{\pi}z)|_{z=b/M}$ ($b \in R_M$) with the Goss L -function.

Lemma 6. Let n be a positive integer and let $M \in A_+$ with $\deg M > 0$. Let f be any \mathbb{C}_∞ -valued function on A/MA satisfying:

- (i) $f(\zeta a) = \zeta^n f(a)$ ($\zeta \in \mathbb{F}_q^*$);
- (ii) $f(a) = 0$ if $\gcd(a, M) > 1$.

Then, we have

$$\sum_{a \in A_+} \frac{f(a)}{a^n} = \left(\frac{\bar{\pi}}{M} \right)^n \sum_{b \in R_M} f(b) P_n \left(\frac{\bar{\pi}b}{M} \right).$$

We now prove Theorem 2. By (2.3), it suffices to prove that $P_n(\bar{\pi}b/M)$ ($b \in R_M$) are linearly independent over K . We assume that

$$(4.1) \quad \sum_{b \in R_M} c_b P_n \left(\frac{\bar{\pi}b}{M} \right) = 0 \quad (c_b \in K).$$

For $a \in (A/MA)^*$, there exists $\bar{a} \in (A/MA)^*$ such that $a\bar{a} \equiv 1 \pmod{M}$. Noting that the Goss polynomial $G_n(X)$ belongs to $K[X]$, we map (4.1) by $\sigma_{\bar{a}} \in \text{Gal}(K_M/K)$ corresponding to $\bar{a} \in (A/MA)^*$. Then, we obtain

$$\sum_{b \in R_M} c_b G_n(\sigma_{\bar{a}}(e(\bar{\pi}b/M))^{-1}) = \sum_{b \in R_M} c_b P_n \left(\frac{\bar{\pi}\bar{a}b}{M} \right) = 0.$$

Using Lemmas 5 and 6, we see that

$$\begin{aligned} \det_{b,c \in R_M} P_n \left(\frac{\bar{\pi}bc}{M} \right) &= \prod_{\chi \in \Lambda} \left(\sum_{a \in R_M} \bar{\chi}(a) P_n \left(\frac{\bar{\pi}a}{M} \right) \right) \\ &= \left(\frac{M}{\bar{\pi}} \right)^{n\varphi(M)/(q-1)} \prod_{\chi \in \Lambda} L(n, \bar{\chi}), \end{aligned}$$

where Λ is the set of all characters of $(A/MA)^*$ given by $\Lambda = \{\chi : (A/MA)^* \rightarrow \mathbb{C}_\infty^* \mid \chi(\zeta) = \zeta^{-n} \text{ } (\zeta \in \mathbb{F}_q^*)\}$. Therefore, it follows from (2.4) that $c_b = 0$ for $b \in R_M$.

4.2. Proof of Theorem 3. For the primitive Carlitz M -torsion point λ_M , we have $\Phi_M(\lambda_M) = 0$. Since Φ_M is defined over K , for any $\sigma \in \text{Gal}(K_M/K)$, we have $\Phi_M(\sigma(\lambda_M)) = 0$. Using the isomorphism $\text{Gal}(K_M/K) \cong (A/MA)^*$, the set of all roots of Φ_M is $\{\rho_a(\lambda_M) \mid a \in (A/MA)^*\}$. Combining (iii) with $K_M := K(\lambda_M)$, we obtain

$$[K_M K_g : K_g] = [K_g(\lambda_M) : K_g] = \varphi(M) = [K_M : K].$$

Hence, K_M and K_g are linearly disjoint over K . By Theorem 2, $P_n(\bar{\pi}b/M)$ ($b \in R_M$), which belong to K_M , are linearly independent over K . Therefore, they are linearly independent over K_g as well. Since g is non-zero, we have $g(b) \neq 0$ for some $b \in R_M$. Hence, using Lemma 6, $\sum_{a \in A_+} g(a)/a^n = (\frac{\bar{\pi}}{M})^n \sum_{b \in R_M} g(b)P_n(\bar{\pi}b/M) \neq 0$. This proves Theorem 3.

4.3. Proof of Theorem 4. We assume that

$$(4.2) \quad \sum_{\chi \in \Lambda} c_\chi L(n, \chi) = 0 \quad (c_\chi \in K).$$

Let $g = \sum_{\chi \in \Lambda} c_\chi \chi$. We note that this satisfies conditions (i) and (ii) in Theorem 3. Using the identity in Lemma 6, we have

$$(4.3) \quad \sum_{b \in R_M} g(b)P_n(\bar{\pi}b/M) = \left(\frac{M}{\bar{\pi}}\right)^n \sum_{a \in A_+} \frac{g(a)}{a^n} = 0.$$

Since $[K_M : K] = \varphi(M)$ and $[K\mathbb{F}_{q^r} : K] = r$ are coprime, K_M and $K\mathbb{F}_{q^r}$ are linearly disjoint over K . Hence, using Theorem 2, $P_n(\bar{\pi}b/M)$ ($b \in R_M$) are linearly independent over $K\mathbb{F}_{q^r}$. Combining (4.2) with $g(b) \in K\mathbb{F}_{q^r}$ ($b \in R_M$), we see that $g(b) = 0$ ($b \in R_M$). Namely, we obtain

$$(4.4) \quad \sum_{\chi \in \Lambda} c_\chi \chi(b) = 0 \quad (b \in R_M).$$

We set $d = \varphi(M)/(q-1)$. Letting $\Lambda = \{\chi_1, \dots, \chi_d\}$ and $R_M = \{b_1, \dots, b_d\}$, we see that the $d \times d$ matrix $(\chi_i(b_j))$ is invertible. Therefore, we conclude from (4.4) that $c_\chi = 0$ ($\chi \in \Lambda$).

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Department of Applied Mathematics
Okayama University of Science, Ridai-cho 1-1
Okayama, 700–0005
Japan
E-mail address: hamahata@xmath.ous.ac.jp

岡山理科大学・理学部 浜畑 芳紀